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What is to believe in a mathematical assertion?

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Abstract. In this brief article we present the following paradox: one cannot assume that mathematicians are trustworthy when they express their mathematical (dis)beliefs, while also maintaining four basic theses about natural and mathematical language. We carefully present the very natural hypotheses on which this paradox is based and then we show how to deduce the paradox from these assumptions. We end by presenting the possible ways in which one can reject the paradox, together with their conceptual implications.

Keywords: Mathematical language, Assertions, Mathematical beliefs, Naturalism in mathematics, Doxastic paradox

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1. Trustworthy astonishment

Cantor once famously wrote to Dedekind: (†) «Je le vois, mais je ne le crois pas».
He was referring to the, then stunning, theorem that the real line has the same cardinality as the real plane, as the three-dimension idealization of the space. Similar expressions of astonishment are common in mathematics: many arguments are labelled as counter-intuitive, or hard to believe[1].
One might take all such remarks as merely rhetorical or idiosyncratic, perhaps akin to aesthetic judgments–yet, *something* is conveyed by them. Our goal is to understand what the philosophical consequences are of taking these expressions of amazement (or confusion) seriously. In particular, from now on we will interpret (†) literally.
The *first working hypothesis* of this paper is that mathematicians sometimes are committed to the existence of a gap between *believability* and *justification*. While justification is typically presented in terms of proof, the process of belief-formation is not exhausted by the demonstrative dimension; as Cantor's example shows, one might not believe in a mathematical statement even in the presence of

[1] In fact, Cantor's story has been taken as paradigmatic of mathematical discoveries which are hard to accept, see Hadamard (1945, p. 61-62).

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a proof. To dispel any doubt, we do not consider such lack of belief as the result of not understanding. In other terms, when a mathematician says that she does not believe in what she sees, this does not mean that she does not understand the corresponding proof.
The *second working hypothesis* of this paper is that the introspection of mathematicians should be trusted. Of course we are not committing ourselves here on a blind trust to the words of mathematicians. We are only assuming a mild position, which we shall call **Sentence Belief Naturalism**:(**SBN**) *Mathematicians are reliable sources for the formation of their mathematical beliefs and disbeliefs.*
One might argue that **SBN** does not hold unrestrictedly. For instance, there might be mathematicians that are not sincere (for whatever reason) when they claim that they believe a mathematical statement to be true or false. Yet, for the sake of the argument we only need that **SBN** holds in a canonical idealization of mathematical practice.

2. The paradox

In what follows, we will show that **SBN** is incompatible with the following theses about assertions and theorems:

1. When we assert a statement 'p', we express a belief that p[2].
2. A theorem is a possible object of an act of asserting.
3. Every mathematician is willing to assert any theorem.
4. For asserting that 'p' is a theorem one only needs a proof of 'p'.

All these theses are natural. The first one is the standard view about one of the things that a speaker *does* when making an assertion (see, e.g., Grice 1957; Searle 1969; Bach and Harnish 1979). The other theses are much less often explicitly defended in the literature; yet, they are often taken as platitudes. Regarding 2., Pagin (2016) offers a mathematical theorem—i.e., that there are infinitely many prime numbers—as a paradigmatic case of what can be asserted[3]. Regarding 3., we acknowledge that a mathematician might not be willing to assert a mathematical statement 'p', but we claim that she will do so if she believes that 'p' is a theorem. For instance, an intuitionist might not be willing to assert the Well-Ordering Theorem, but this only means that she rejects the proof of the theorem (thus claiming that in fact the Well-Ordering Theorem should not count as a theorem). Regarding 4., to accept that a statement (appearing, e.g., in a paper) is a theorem, the mathematical community requires nothing more and nothing less than its proof.
Proposition 2.1.*Theses (1)-(4) cannot stand together with SBN.*
Proof. Let 'p' be a proven mathematical fact that is not believed by some mathematician **M**. Since 'p' has a proof then, by (4), 'p' is a theorem. It follows from (2) that 'p' is a possible object of an act of asserting and from (3) that **M** is willing to assert 'p'. But if **M** asserts 'p', then by (1) **M** is expressing her belief that p. Hence, **M** is wrong with respect to her disbelief of p—and this contradicts **SBN** (relatively to M).

[2] By 'p' we refer to a statement signifying the proposition p.
[3] For a detailed analysis of theorems as assertive speech acts, see Ruffino, San Mauro, and Venturi 2020.

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3. Possible ways out

Apparently, there are four possible ways of escaping the paradox. All of them come with a theoretical price:

I. *By asserting a theorem 'p', one does not express one's belief that p*[4]. To defend this stance, one has to negate either (1) (thus saying that asserting a theorem one does not express a belief in the theorem) or (2) (thus saying that theorems are not possible objects of an act of asserting).

II. *There are theorems that some mathematicians are not willing to assert.* To defend this stance, one has to negate (3) (but then one has to offer reasons why a mathematician would not assert a theorem—which is arguably against mathematical practice).

III. *To conclude that 'p' is a theorem one needs more than just a proof of 'p'*[5]. To defend this stance, one has to negate (4) (thus going against the common behaviour of the mathematical community, which just publishes proofs, rather than, say, beliefs).

IV. *Some mathematicians are wrong in saying that they do not believe a given theorem.* To defend this stance, one has to negate **SBN**.

To sum up, by accepting **SBN** (i.e., accepting that mathematicians are to be taken at face value when talking about their mathematical beliefs), one has to endorse at least one of the following (rather undesirable) views: some assertions do not express beliefs; some theorems cannot or would not be asserted by some mathematicians; some theorems are such in virtue of more than their proofs.

4. Dispelling the paradox

We noted that, by assuming a literal interpretation of (†), onemakes **SBN** incompatible with four widely accepted theses about assertions and theorems. So, does it follow that (†)and similar expressions of mathematical astonishmentare to be taken figuratively or tongue in cheek? Not necessarily. In fact, we suggest that there *is* a way of saving a literal interpretation of (†) and *not* generating the above paradox.

Our suggestion is that the two instances of "it"in (†) have different meanings. The first "it" (I see *it*) refers to the formal proof that the real line has the same cardinality as the real plane—hence, it is based on the formal notion of cardinality to which the theorem applies. The second "it" (I don't believe *it*) is rather based on the pre-formal notion of cardinality that Cantor aimed at capturing with this formal notion. So, Cantor's astonishment was rooted in the difficulty of conceiving a formal rendering of cardinality which does not preserve dimensionality. If this interpretation is correct, we can assume that Cantor is reliable with respect to his beliefs (especially the mathematical ones) and, therefore, that **SBN** is not contradicted. On the other hand, the possibility to separate the mathematical and the conceptual content of Cantor's belief allows one to stop the insurgence of the paradox at its very start. Specifically, Cantor is not doubting neither the correctness nor the meaningfulness of his theorem, but he is only expressing a distrust of the possibility to formalize the notion of cardinality through the well-known principle of equinumerosity.

[4]But perhaps only one's belief in the proof of 'p'.
[5]E.g., one might also need a belief that p.

As any other solution to the above paradox also this one has a cost, which, however, we feel more inclined to accept. This consists in accepting that mathematics, sometimes, has a content which is independent from its formalization. What is relevant for this discussion is that a naturalist perspective towards language and beliefs leads to an impasse whose more acceptable way out consists in a fruitful separation of levels between form and content, between symbols and ideas. We take this as a confirmation that a linguistic approach to mathematics can have useful applications to the philosophy of mathematics.

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