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# Numerical cognition and philosophy of mathematics. Dehaene's (neuro)intuitionism and the relevance of language

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**Abstract** Can investigations carried out in the field of cognitive science about numerical cognition shed light on the issues traditionally raised by philosophers of mathematics? What kind of relation exists between the current neuroscientific research and the philosophical reflection on mathematics? This paper critically explores the answers to these questions provided by the French neuroscientist Stanislas Dehaene, identifying the framework of a possible future collaboration between these disciplines.

**Keywords:** Language and numbers, Cognitive science of mathematics, Numerical cognition, Innate knowledge; Mathematical knowledge

## 0. Introduction

One of the most interesting and productive fields of research in cognitive sciences over the latest years has been the one concerning experimental studies on numerical cognition. Some neuroscientists decided to take on the task of shedding some light on many of the traditional issues addressed by the philosophy of mathematics, for example the understanding of the concept of number, the nature of mathematical knowledge, the link between mathematics and natural sciences, etc. Yet, it is often possible to notice a strong opposition between people believing that cognitive sciences on their own can explain everything in mathematics, and those claiming that the research carried out by scientists and neuroscientists cannot contribute to the debate but only provide insights for an additional and thorough theoretical and philosophical analysis.

Fortunately for those who believe that interdisciplinary dialogue and debate are crucial and that all different scientific branches could – and should – be involved in a dialogue in order to provide a complete and detailed picture of human sciences, the opinions presented above are not dominant or prevailing. On the contrary, in the wide panorama of philosophy of science, it is possible to notice the emergence of an interdisciplinary effort to provide relevant answers to the different branches of mathematical knowledge through a series of attempts that combine theoretical approaches with empirical data. The logic behind these efforts is that philosophy and science can and shall establish a dialogue because, on one hand, science needs theories and hypothesis that could provide a useful basis to interpret and understand the empirical data at a general level, while on the other hand the philosophy of mathematics needs indeed to be underpinned by objective considerations that could avoid its detachment from reality.

This enquiry focuses on the search for a “third option”; an alternative to both a return to an independent philosophy of mathematics and an uncritical application of the idea that the data stemming from cognitive science research are enough to single-handedly explain every interesting fact about mathematics and its constructions. From this perspective, philosophy would take on the daunting task of describing and explaining the specificities and similarities of different types of advanced mathematical explanations.

Looking with interest at this kind of debate, this paper addresses one of the most remarkable breakthrough made in cognitive science by French neuroscientist Stanislas Dehaene; mathematics as an inborn skill of living organisms.

Despite being one of the most skilful and authoritative researchers when it comes to finding solutions to the main questions concerning the human mind, the biological organisms, or the cognitive architecture underpinning human understanding of the fundamental concepts of mathematics, Dehaene seems unfortunately unable to answer the philosophical questions raised by his research and, in general, by the discoveries made over the years in the field of cognitive sciences. As this paper will illustrate, Dehaene was convinced that he could provide an empirical support to the ideas postulated by Kant and Brouwer, and he stated on different occasions that intuitionism was the philosophical school best suited to explain recent discoveries. On the contrary, the hypothesis put forward in this paper is that Dehaene’s discoveries are not only in opposition to the theories of those philosophers but that even the very discovery of some kind of approximate basic form of mathematics goes against many of the traditional philosophical theories put forward by intuitionists. The main objective of this paper is to suggest the idea that the contribution of cognitive sciences to mathematical reflection would become more useful if the peculiarities of the theories advanced by philosophers of mathematics were taken into full account.

## 1. Dehaene's (neuro)philosophical thesis

In recent years, due to the tremendous progress occurred in cognitive sciences, many of the questions concerning the nature of mathematical knowledge have transcended the traditional boundaries of philosophy and they have become the object of the research carried out by neuroscientists, neurobiologists, and cognitive psychologists. An example, among many others, is provided by the studies of neurophysiologist Stanislas Dehaene (1997, 2011), who postulated the existence of two different cognitive systems relating to mathematical skills:

1) The first system is not based on symbols and it is approximative; it is based on the estimation of quantities; and it involves both a simple process of comparison and a series of basic arithmetical operations like addition and subtraction[1].

[1] In this instance, the ability shown by children and animals in experiments where they were asked to perform additions and subtractions recalls – although approximately – the concept of ‘numerical expectations’ which is based on object trajectory, not on object identity. As Dehaene explains: «Thus, babies may not know whether 2 + 2 is 3, 4, or 5, yet they may still be surprised if they see a scene

2) The second system is based on symbols and it is language- and culture-dependent; it is typical of adults, and it is founded on the ability of counting, therefore on a numerical system and on all arithmetical operations.

The first system is culture – and language – independent. It refers to the technical concept of numerosity, i.e. the simple perceptual estimation of the size of different sets of objects and the ability to understand whether two sets of objects are equal or not[2]. This ability has been found both in infants and animals and it is called 'proto-numerical skill' or 'pre-numerical skill'. This skill follows two main principles: numerical distance and magnitude. The former dictates that the ability to discriminate between two numerical quantities improves as the distance between the two sets increases[3]. The latter, instead, establishes that when the distance between two sets is the same, the ability to estimate the biggest one becomes worse as the number of elements that they contain increases[4].
The second system instead is culture-dependent and based on the knowledge of symbols and rules. Therefore, it is strictly linked to language and hence it is typical of adults. This system allows for the 'canonical' use of numbers[5], in other words the use of a number in a precise way and referred to an exact quantity.
Starting from these considerations, over the last two decades, neuroscientists like Dehaene and many others developmental psychologists carried out a series of studies with the aim of finding the 'anti-Piagetian'[6] empirical proof that human beings are born with a 'number sense', a label used to describe the existence of a 'mental organ' or a set of brain circuits also existing also in other species which would work like an *accumulator* – an approximate counting device – hence helping in estimating, memorising and approximately comparing numerical quantities (DEHAENE 1997, 2011).
As Dehaene explains in the preface to the second edition of his book, *The Number sense*:

> Fifteen years have elapsed since I proposed my number sense hypothesis – the peculiar idea that we owe our mathematical intuitions to an inherited capacity that we share with other animals, namely, the rapid perception of approximate numbers of objects. How does such a preposterous notion hold up after fifteen years of intense scrutiny? Surprisingly well, I would say. Number sense is now recognized as one of the major domains of human and animal competence, and its brain mechanisms are constantly being dissected in increasing detail (DEHAENE 2011: 237).

suggesting that 2 + 2 is 8» (DEHAENE 2011: 45).

[2] Through estimations of relative numerosity, for example, it is possible to establish that the set 'X' contains more elements than the set 'Y', without necessarily having any knowledge about the absolute value of the elements that 'X' and 'Y' contain.

[3] For example, in this way it is easier to discriminate between two sets containing respectively two and five objects, compared to two sets containing respectively four and five objects.

[4] In other words, it is more difficult to discriminate between two sets containing respectively ten and eleven elements, compared to two sets containing two and three elements, despite the fact that the distance between the elements of the two sets is always one.

[5] The expression 'canonical use of numbers' refers to the use of a number in a precise manner to indicate exactly the quantity represented by the number, in other words what corresponds to the 'cardinality' of a finite set.

[6] These experimental studies question the ideas postulated by Piaget (1952), according to whom the structures of general intelligence and the development of numerical skills are indivisible.

Specifically, a thesis Dehaene puts forward on several occasions is that humans are born with different intuitions of number, sets, continuous quantities, interactions, the geometry of space. He affirms that:

> From grid cells to number neurons, the richness and variety of the mechanisms used by animals and humans, including infants, to represent the dimensions of space, time and number is bewildering and suggests evolutionary processes and neural mechanisms which may universally give rise to Kantian intuitions (DEHAENE, BRANNON 2011: IX).

Dehaene contends that his research program might be easily defined as a 'Kantian research program', because it is aimed at understanding how the intuitions that make possible all experience in the world are created, which neurones are involved with these intuitions, and how they might be modified over time through education and learning (DEHAENE, BRANNON 2011).

It seems therefore that Dehaene considers the '*a priori* knowledge' in its Kantian meaning as a form of 'innate' knowledge, because it exists from the moment an organism is born[7]. In this context, therefore, 'innate' refers to a 'potential' ready to be developed, provided that the environment is favourable for its development, highlighting in this way the mutual influence existing between biology and environmental factors.

Nevertheless, the term 'innate' is often mistaken for 'unchangeable' and this is why even Dehaene uses it as synonym of *a priori*. Despite this fact, contending that '*a priori* knowledge' is 'innate knowledge' strongly limits the scope of '*a priori* knowledge', since this kind of knowledge is supposed to be not just the exclusive result of biological evolution as much as 'innate knowledge', but also the result of cultural evolution. This consideration clarifies that '*a priori* knowledge' represents a wider form of knowledge compared to 'innate knowledge'.

Contrary to the concept of '*a priori* knowledge' of Kant[8], the 'innate knowledge' defined by Dehaene is not created independently of all experience, because it is obtained by formulating hypotheses based on a series of premises that could be the product of experience. Besides, the premises and conclusions of these hypotheses are plausible only if compared to experience. Furthermore, the same knowledge does not hold universally, since in future (according to some possible evolutional law) some exceptions might emerge. At the same time, this knowledge is not intrinsically necessary, but it is contingent, since it could result compatible or incompatible with future data. Finally, it is not certain in itself, because there is no guarantee that examples invalidating it won't be found in future.

As a matter of fact, Dehaene is in some way forced to admit that:

> This research is stimulating innovative research focusing on the search for representations of space, time and number inherited from evolution. We must, however, acknowledge that the word 'innate', meaning 'independent of

[7] The idea that '*a priori* knowledge' is the same of 'innate knowledge' represents a widespread belief, even among philosophers of science. For example, Popper affirmed that «*a priori* knowledge is that type of knowledge that an organism has before the experience of senses, it is an innate knowledge» (POPPER 1990: 46).

[8] According to Kant, '*a priori* knowledge' is 1) created independently from all experience; 2) rigorously universal, with no exception; 3) intrinsically necessary and 4) certain in itself.

> experience', is an idealization which will ultimately have to be replaced by detailed research into the underlying genetic and developmental mechanisms (*Ivi*: X).

Obviously, this should not be interpreted according to the Kantian definition of '*a priori* knowledge'. Asserting that '*a priori* knowledge' – in its Kantian meaning – is experience-independent is a logical remark, not a descriptive one. Therefore, it does not mean that knowledge can be obtained independently from a concrete experience. It does not represent an observation about how reality works, but it rather suggests that knowledge precedes logically the possibility that every specific experience is going to be shaped according to its actual shape. According to Kant, '*a priori*' represents indeed a synonym of 'transcendental' and it is related to the possibility of experience, not its more or less accurate description.
Therefore, the way in which Dehaene considers mathematics as 'innate knowledge' is quite different to the views of Kant, who focused on the character of absolute ineffability of mathematics as an absolute and certain '*a priori* knowledge'.

## 2. Dehaene and Brouwer's intuitionism

Yet, some authors (LONGO, VIAROUGE 2010) have advanced the idea that Dehaene's starting postulations are not based on the work of Kant but instead they seem to be more in line with the ideas of Luitzen Egbertus Jan Brouwer, the main representative of the foundational intuitionist school of the twentieth century. On the other hand, in a document that Dehaene prepared in 2006 as introduction to his course he explained:

> The position I am defending (in this course) and which you can qualify as intuitionist, does not belong to any of these fields [Platonist and formalist]. It postulates that the cognitive foundations of mathematics must be sought in a series of fundamental intuitions of space, time, and number shared by many species of animals and which originate in a distant past where these intuitions played an essential role to survive. Mathematics is built on the formalisation and creation of a conscious relationship among these different intuitions. This position is close, but not identical, to the mathematical intuitionism of Brouwer (DEHAENE 2006: 277-278).

As a matter of fact, Dehaene claims that he has enough evidence – gained through research and several experimental activities – to reject platonism and formalism, and agree with intuitionism, especially because of the constructivist perspective held by this approach. As he writes:

> Classical mathematics are based on an intuition of the dichotomy between truth and falsehood (and as such, as noted by Brouwer, they indeed run the risk of going beyond our intuitions about finite and infinite sets). Brouwer, on the contrary, adopts the primacy of finite constructions or reasonings as a fundamental principle. In the final analysis, his version of mathematics, although it is sometimes called "intuitionism", is certainly not more intuitive than others – it is merely based on a partially distinct set of intuitions. In this framework, then, what remains to be explained is how, on the basis of the innate categories of their intuition, mathematicians elaborate ever more abstract symbolic constructions (DEHAENE 2011: 228).

Generally speaking, it looks like Dehaene constantly refers to classical intuitionists and especially to Brouwer because he is particularly attracted by an approach that – contrary to logicist views – can be framed in the tradition of thought that joins thinkers from Kronecker to Poincaré, and which aims at recovering the identity of mathematics as a science independent of logic. From this perspective, mathematics becomes a science with its own content, which is created directly without the mediation of intuition. Dehaene seems to be particularly interested in Brouwer's scientific work and therefore he refers often to it, recalling the ideas dating back to the beginning of 1907, especially his theory on a clear separation between thought and language. This separation of the language of mathematics from mathematics is the subject of the *First Act Of Intuitionism:*

> Completely separating mathematics from mathematical language and hence from the phenomena of language described by theoretical logic, recognizing that intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the twoity thus born is divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is this common substratum, this empty form, which is the basic intuition of mathematics (BROUWER 1981: 4-5).

According to Brouwer, mathematics is the product of the human mind, which is based on Primordial Intuition. When dealing with Primordial Intuition (hereinafter PI), the subject becomes aware of the discrete elements in time, in other words he becomes aware of two discrete entities, one belonging to the present and the other belonging to the past. In this way, as it is described by Brouwer: «By a move of time a present sensation gives way to another present sensation in such a way that consciousness retains the former one as past sensation» (BROUWER 1948: 1235).

Primordial Intuition can be endlessly repeated; it depends only on the free will of the subject, thereby producing sequences of increasingly complex mental objects simply as repetition of the primordial act. With a twoity, it is possible to build a threeity; with a threeity it is possible to build another construction, and so on. Therefore, according to Brouwer, all numbers – ordinals, natural and other – are constructions obtained from reiterations of PI: «This intuition of two-oneness, this ur-intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers» (BROUWER 1912: 12)[9].

Brouwer's PI is identified with the intuition of time: our consciousness recognises two distinct moments, the past and the present. PI is therefore what allows subjects to observe two discrete elements of the temporal continuum, or rather it is what allows individuals to notice change (BROUWER 1907).

When referring to time, Brouwer claims himself to be inspired by Kant[10] and, in particular, from the idea that the basic intuition of mathematics that creates natural numbers is a temporal intuition and therefore the apriority of time qualifies the

[9] Indeed, it is always possible for a person to build a greater number, because the number of possible repetitions of PI is unlimited; the only limit is represented by the free will of the creator.

[10] The construction of number through temporal intuition in Kant is much more complex compared to the one presented by Brouwer. Besides, Brouwer considers temporal intuition as creative, and it is not limited to avoid concepts void of content as in Kant's theory.

properties of arithmetic as synthetic *a priori* judgements. Brouwer is therefore interested in an *a priori* concept linked to both Kantian definitions of 'independent from all experience' and of 'necessary condition for science'[11]. Furthermore, Brouwer is convinced as Kant that the fundamental duty of the philosophy of mathematics is to investigate the foundations on which the certainty of mathematics rests.

In particular, Brouwer contended that it was necessary to develop a new type of mathematics, alternatve to classical mathematics, in which only objects and methods given by intuition were to be used. Only in this way, it was possible to assess entirely the reliability of mathematical reasoning, with a reasoning process starting from axioms based on intuition and then developed following deductive inferences based on intuition. Considering what mentioned above, in the view of Brouwer intuition represents the only source for an absolutely certain knowledge of mathematics, because intuition is able to make us flawlessly understand mathematical truths.

The primordial, basic, and immediate intuition postulated by Brouwer leads Dehaene to the conclusion that intuitionism is the theory that, from a philosophical point of view, best explains the current research on the links between arithmetic and the brain[12]. He states that:

> What, indeed, did the preceding chapters reveal about this natural number sense?
> - That the human baby is born with innate mechanisms for individuating objects and for extracting the numerosity of small sets.
> - That this "number sense" is also present in animals, and hence that it is independent of language and has a long evolutionary history.
> - That in children, numerical estimation, comparison, counting, simple addition and subtraction, all emerge spontaneously without much explicit instruction.
> - That the inferior parietal region of both cerebral hemispheres hosts neuronal circuits dedicated to the mental manipulation of numerical quantities.
>
> Intuition about numbers is thus anchored deep in our brain. Number appears as one of the fundamental dimensions according to which our nervous system parses the external world (DEHAENE 2011: 227).

Nevertheless, after further analysis, it is clear that the ideas outlined by Brouwer clash with the findings of the French neuroscientist.

On the other hand, according to Dehaene, the term 'mathematics' has a narrower connotation that includes only the processes and representations involved in tasks like counting, addition, multiplication, comparison of numbers, etc. In other words, Dehaene's conception of mathematics is closer to the concept of common sense. As a matter of fact, according to Dehaene, exact mathematics is symbolism, therefore a type of language. In other words, it is what in the mystical view of Brouwer was considered as something to avoid because it was linked to activities aimed at the outside world. Contrary to what Dehaene contends, Brouwer supports the idea that language cannot guarantee mathematical exactness. According to Brouwer, the truth and validity of mathematics must be found in the mental process. For Dehaene, the

[11] As it is known, after some years, Brouwer decided to abandon the idea of the apriority of space and embraced exclusively the apriority of time, considering the latter as the *a priori* form of every change and, consequently, the reference to Kant disappeared in his following works.

[12] To analyse further the philosophical issues linked to intuition and how it is straightforward, immediate, and hence non-inferential and not related to the availability of 'core knowledge', please refer to Cellucci (2013).

language-less feature becomes the fundamental basis of elementary, approximative mathematics, shared by humans and other animals. While both authors share the need of a conception of mathematics not based on language, for Brouwer this need involves the entire realm of mathematics because of a series of purely human reasons (mystical needs), while for Dehaene it involves only approximate mathematics.
Therefore, even if Dehaene seems to be pretty convincing in the field of cognitive neuroscience and in the identification of the cognitive structures that provide basic mathematical skills, he seems to be ill-equipped – as highlighted in the first paragraph in connection with the concepts of 'inborn skill' and '*a priori*' – in the philosophical vocabulary that has been historically used by the intuitionist foundational school and its main representative, J. Brouwer. Nevertheless, in the search for a connection between Dehaene's experimental findings and the ideas outlined by classical intuitionists, the right place to start wouldn't be Brouwer's ideas, rather those of one of his pupils; George François Cornelis Griss.

**3. The theory put forward by George François Cornelis Griss**

The philosophical heritage of Griss is contained in a small book where he presents his idea of philosophy of mathematics and entitled *Idealistische Filosofie*, and in a philosophical article entitled *Sur la negation dans les mathematiques et la logique* where he outlines how it could be possible to build a 'negation free' intuitionist mathematics. Even Griss (1946) labelled his own philosophy of mathematics as an effort of 'empirical idealism', because he wanted it to be a synthesis between Hegelian idealism (or at least, what Bolland described as Hegelian idealism) and Brouwer's intuitionism. In order to do so, Griss explained that his philosophy of mathematics did not support the existence of anything in itself, as realism did, but it recognised the importance of experience, as empiricism did and contrary to the theories of absolute idealism. He explained that his philosophy of mathematics, contrary to absolute idealism, was not dependent on the validity of deduction, and that contrary to empiricism it was not based on apriority. Griss affirmed that apriority, the 'original information', was seized by the consciousness when it reached its wholesomeness, making it possible for the subject to distinguish itself from the object, since one element has no sense without the other:

> In consciousness the "I" realizes a separation between itself and the other (subject and object) and therefore the unity of both. More detailed: The 'I' detects an object and realizes (is aware of or understands), that this is only possible because of a division between subject and object, in which however the object does not get loose from the subject, but keeps interconnected, involved with each other. The separation is not an absolute division, the unity not unbreakable. (English translation from the Dutch original, GRISS 1946: 14).

According to Griss, the *a priori* is the basic condition of all experience and it can be viewed from three different perspectives: focusing on the subject and object (mysticism), considering both aspects in their connection (philosophy), or focusing on the entities resulting from their separation and on their mutual relations (mathematics). The mathematics in the *Weltanschauung* of Griss divides subject and object, allowing to produce new elements in which there is mathematics (object) only because there are mathematicians (subject). It is therefore clear that, according to him, they cannot exist independently.

The separation between subject and object is particularly important considering our current purposes, because it leads Griss to the conclusion that mathematics in this stage is 'vague'. According to Griss, this vagueness is essential to the development of mathematics because it is the source of infinite sequences, in the sense of indefinite sequences in which their succession and end is unknown because it depends on the subject and not on general criteria. At the same time, even if necessary, this vagueness appears to be intrinsic, because in the 'original information' subject and object were united, therefore the more the subject is separated from mathematics in order to produce it at a mental level, the more inaccurate it becomes. This concept is in direct contrast to Brouwer[13], who affirmed that mathematics was accurate only during mental experience and that it became inaccurate when it moved to the outside world, because language (and memory) could not guarantee the truth of inner mathematics (and this led to all the criticism levelled against formal mathematics, etc.). According to Brouwer's pupil, Griss, mathematics is vague in the human mind, since it is based on the 'original information' of the subject-object division. Therefore, accuracy is only possible from a formal point of view, as Griss contends that signs (as opposed to their referents) do not vary over time. Therefore, language is the only medium that can produce exact mathematics.
As a matter of fact, he states:

> How the thoughts in mathematics are provided is in principle indifferent, but besides in simple cases only in writing printed signs and symbols appear to be useful. On the other hand it is impossible to abstract all content, unless one understands here that, that content is indifferent (according the criticism of Berkeley on abstract concept 1). Some intuitionists, among which Brouwer himself, think they reach the absolute purity of mathematics or strictness with that. For example do we write the numbers 1, 2, 3, … than these content are the representation of the written signs (and not the representation of amounts, because there is nothing to be counted). 2). Does the separation of these representatives succeed, than this gives the appearance that complete preciseness can be reached. They even blame the used tools (the written signs in this example) in case of failure, instead of to search the cause in thinking itself; whereas rather this thinking without the tools would be a lot less exact. Also on the language or the ability to remember. 3). Becomes the not-exactness known. Exactness however is not so far as reachable, that of a total separation of mathematical objects in thinking is out of the question; by the way thinking of two object in the same time would be in that case impossible. Just because separation is not completely, the mathematical thinking is possible, but also making mistakes, that means the inseparable, inevitable (English translation from the Dutch original, *Ivi* 1946: 22-23).

Hence, in his book, Griss provides a definition to a series of concepts (e.g. the ambiguity of mathematics, the role of language, etc.) in line with the ideas supported by Dehaene. The common basis of their rationale is that language is the only means to provide certainty to mathematics, while non-language-based mathematics is approximative and not exact, subject to errors.

[13] As it is well known, another of Brouwer's pupils, Arend Heyting, concluded that the ideas of his former teacher did not constitute the best and most correct approach to bring forward the philosophy of mathematics, arguing that generally speaking it was possible to achieve complete consensus only on formal questions (HEYTING 1958).

Dehaene indeed supports the idea that the exposure to language and, more generally, to culture allows human beings to acquire accurate and certain mathematical skills. As he explains:

> Humans, however, have been endowed by evolution with a supplementary competence: the ability to create complex symbol systems, including spoken and written language. Words or symbols, because they can separate concepts with arbitrarily close meanings, allow us to move beyond the limits of approximation. Language allows us to label infinitely many different numbers. These labels, the most evolved of which are the Arabic numerals, can symbolize and discretize any continuous quantity. Thanks to them, numbers that may be close in quantity, but whose arithmetical properties are very different, can be distinguished. Only then can the invention of purely formal rules for comparing, adding, or dividing two numbers be conceived. Indeed, numbers acquire a life of their own, devoid of any direct reference to concrete sets of objects. The scaffolding of mathematics can then rise, ever higher, ever more abstract (DEHAENE 2011: XIX).

Therefore, despite the fact that Dehaene was (rightly) convinced that a good part of basic numerical cognition is based on non-verbal systems, when it comes down to explaining how complex numerical concepts that are not immediately perceivable are learned, Dehaene states that those concepts cannot be acquired without the use of language. This leads to the conclusion that Dehaene does not question the idea that language plays a big role in the development of mathematical skills, but rather how big this role really is.

## 4. Numbers and numerosity

Considering what has been said above, it is clear that the stance defended by Dehaene – which he defines intuitionist – is completely different to the traditional philosophical ideas developed in the framework of intuitionism. In fact, Dehaene's hypotheses merely underline the existence of a kind of natural, inborn, biologically-founded mathematics shared between human beings and animals and existing already in newborns in a prelinguistic stage of their development. Dehaene argues indeed that, through natural selection, nature shaped animals and living beings that could perform actions and natural mathematical calculations that are helpful in the development of behaviours crucial to their survival.

Dehaene's arguments deals therefore more with the concept of numerosity, a term used to describe the number sense and, in particular, the sense used to identify the size of a set of objects – which, as we have already seen, is subject to the distance and size rule – and not the size of numbers. Numerosity indicates therefore a simple perceptual evaluation of different sets of objects and the connected ability that deals with the comparison with larger or smaller sets. Hence, the concept of numerosity merely refers to a cardinality that is not based on symbols. In opposition to this, in mathematics, numbers represent abstract entities with specific proprieties (e.g. continuity in the case of real numbers). Furthermore, numbers are represented by specific oral and written symbols and they can be used in calculations. Finally, according to their proprieties, numbers are classified as natural, real, imaginary, negative, integer, rational, irrational, complex, and so on.

During their development, human beings learn the symbolic use of numbers in their different forms and in relation with other cognitive skills. In opposition to this, the

animals and newborns in a prelinguistic development stage on which Dehaene focuses in his experiments, showed the ability to perform elementary calculations that are substantially linked to a simple approximative interpretation of numerosity.
At the core of this reasoning lies the notion that the development of a concept of number seems needed to have a word or a symbol that can be used to refer to its concept. In the philosophical tradition, as it is well known, this idea was strongly supported by Aristotle, who stated that «number is perceived by the negation of continuity» (ὁ δ' ἀριθμὸς <αἰσθανόμεθα> τῇ ἀποφάσει τοῦ συνεχοῦς) (*De Anima*, 425a 19). Nevertheless, as Franco Lo Piparo highlighted, in this sentence the word ἀπόφασις was literally translated with the word 'negation', putting therefore in second place the Aristotelian reference to the intrinsically linguistic nature of the practice of the negation of continuity. In other words, as Lo Piparo explains:

> It should not be seen as the 'elimination' of something but as a linguistic act that states that something is different: ἀπόφασις comes from ἀπόφημι, which means 'I say no', 'I say that this is not that'. A number, because of its 'multiplicity of units', represents discontinuous entities perceivable following a cognitive operation that, describing the units used to identify them, differentiate them. A number therefore requires a linguistic negation act (LO PIPARO 2014: 186).

To provide even more clarity on the subject, Lo Piparo explains also the difference outlined by Aristotle between 'unit' and 'point', which is useful also for the purposes of this paper. Lo Piparo explains that:

> To highlight even more the linguistic ontology of a number, Aristotle stresses the importance of avoiding confusion between *unit* and *point*. While a *unit* is a result of a linguistic operation ("the negation of continuity"), a *point* is a physical sign of a concrete act. "A point and every division and whatever is indivisible in this way is made clear like a *privation*" (*De An*., 430b 20-21). In other words: a *point* is the product of a subtraction from a physically well-delimited entity. This does not necessarily require a linguistic intervention. A point exists in the frame of a physical space, while a unit exists merely at a cognitive-linguistic level (LO PIPARO 2014: 187).

Therefore, following the path charted by Lo Piparo and his reference to Aristotle, a unit must be regarded as a linguistic operation necessary to the development of the quantification skills that are at the basis of arithmetical knowledge. Using the same words of Lo Piparo, numerical activities are in fact «operations that require language. A *unit* can be used for calculations only if expressible, i.e. expressibility is a condition for its arithmetical existence» (LO PIPARO 2014: 186).
Going back to the topic at hand, what is remarkable is that Dehaene seems to address the issue following the same rationale. He states for example that:

> Suppose that our only mental representation of number were an approximate accumulator similar to the rat's. We would have rather precise notions of the numbers 1, 2, and 3. But beyond this point, the number line would vanish into a thickening fog. We could not think of number 9 without confusing it with its neighbors 8 and 10. Even if we understood that the circumference of a circle divided by its diameter is a constant, the number π would only be known to us as "about 3." This fuzziness would befuddle any attempt at a monetary system, much of scientific knowledge, indeed human society as we know it. How did *Homo sapiens* alone ever move beyond approximation? The uniquely human

> ability to devise symbolic numeration systems was probably the most crucial factor. Certain structures of the human brain that are still far from understood enable us to use any arbitrary symbol, be it a spoken word, a gesture, or a shape on paper, as a vehicle for a mental representation. Linguistic symbols parse the world into discrete categories. Hence, they allow us to refer to precise numbers and to separate them categorically from their closest neighbors. Without symbols, we might not discriminate 8 from 9. But with the help of our elaborate numerical notations, we can express thoughts as precise as "The speed of light is 299.792.458 meters per second" (DEHAENE 2011: 79).

The real problem is the fact that Dehaene and, in general, neuroscientists dealing with numerical cognition are interested in 'small' and 'common' natural numbers that often do not fare over 10. In fact, there are no numerical cognition studies that address natural numbers such as 299,792,458 even if, mathematically speaking, they belong to natural numbers as much as the number 4. In some cases:

> there is a risk of unwittingly ascribing 'numerical' properties, such as order or operativity to a far simpler ability to discriminate between stimuli. The mosquitofish is no mathematician. Yet such an ascription leads to the teleological argument that thousands of species, from fish to humans, have full-fledged 'number' representations as a result of biological evolution. Confusion must be avoided (NUNEZ, MARGHETIS 2014: 1).

In addition to this, to make things even more obscure, Dehaene loosely replaces the term 'number' with 'numerosity', especially in the description of the experimental contexts where – in order to prove that even without language great mathematical skills can be developed – the author employs the term 'points' instead of 'numbers', going against the Aristotelian 'advice' mentioned in the previous paragraphs.
This happens for example in the description of the experiments carried out by the French neuroscientist on the Mundurukú population that lives in the Amazon forest, whose language has only few words to identify numbers (up to five, then their numeration system is based on the concepts of few – 'adesu' – or many – 'ade'). In order to show that, despite of the lack of education and a rich vocabulary for numbers over 5, the indigenous Amazonian Mundurukú population own a well-developed number sense, Dehaene and his team involved 55 Mundurukú in a series of quite advanced tests, in which the experts asked the participants to perform two main tasks: compare two sets of points, and add and compare different sets of points. In the first test, the subjects involved were simultaneously presented with two sets of points placed one on the side of the other: the set on the left was black, while the one on the right was red. Then, they were asked to determine which set contained more points and the final results showed that 70.5% of the subjects provided the correct answer. In the second test (addition and subsequent comparison of sets), subjects were presented with an empty jar that was then shown in vertical position, in order to make the subjects aware of the fact that what was in the jar could not fall out. Then, from the upper part of the screen, two sets of points started falling in the jar. The two sets of points did not appear simultaneously on the screen but they followed each other, with each set moving for 5 seconds. Finally, on the right of the jar, a third set of points appeared and the participants were asked to determine whether there were more points inside the jar or outside of it.

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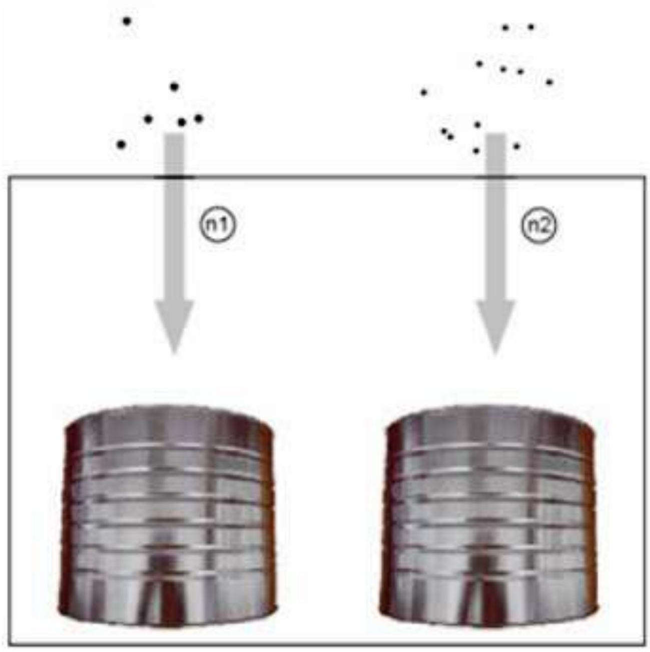


**Fig. 1** Representation of the experimental setting for assessing the ability to compare sets of points: two sets of points falling in empty jars (DEHAENE 2011: 262)

The Mundurukú participants in the experiment performed quite well, with an accuracy rate in the answers provided that reached 80.7%. Nevertheless, commenting these experimental data, Dehaene stated that:

> On both tasks, the Mundurukú failed to calculate the exact result. They performed relatively well with numbers below three, but they failed increasingly often as the numbers got larger, not faring over 50% correct as soon as the initial number exceeded 5. A mathematical model showed that they performed exactly as one would expect, given their capacity to approximate – they *approximated* an operation as simple as 5–3! (DEHAENE 2011: 262-263)

In this way, even if the experiments of Dehaene and his team showed that the Mundurukú population could master some arithmetical concepts such as quantity, together with the ability to determine which set is larger or smaller, or to perform small approximation tasks and despite they highlighted the fact that a symbolic numeration system is needed in order to go beyond this approximative system and perform exact calculations, they ended up contradicting their own conclusions, since their methodology described tasks that the Mundurukú participants performed on 'points' by employing numbers such as 5, 3, 2. On the contrary, the experiment showed that with the exception of 1 and 2, all numbers were used by the indigenous people in relationship to a wide range of approximate quantities rather than in relationship to a precise number (e.g. the word five, which can be translated as 'hand' or 'fist', was used for 5 but also for 6, 7, 8, or 9 points).

In conclusion, even the experiment with the Mundurukú people showed that advanced numerical skills cannot be developed without a sound numerical jargon. The ability to use big numbers is indeed featured exclusively by the cultures – like our own – that have a wide range of words to describe the names of exact numbers and the syntactical rules needed to combine them and produce an endless quantity of names for numbers.

## 5. Conclusion

Many neuroscientists share the opinion that only cognitive science – or better said, the interdisciplinary study of mind/brain supported and backed by a multitude of studies in the field of neuroscience and cognitive psychology – and not philosophy can provide answers to the questions concerning the true nature of mathematics. Nevertheless, the main concern raised when researchers are tempted to include empirical data in mathematical studies is that this approach could lead to a complete or partial loss of the normative dimension that is considered so important in mathematics, submitting to a descriptive epistemology that could turn mathematics into a 'branch of psychology and, therefore, natural science'.
As a matter of fact, differently from many of his colleagues, Dehaene dodges this criticism by taking a clear stand in the philosophical debate concerning mathematics and supporting the ideas of mathematical intuitionism.
Therefore, despite the fact that Dehaene has indeed the merit of finding a connection between the output of his experimental research and theories in the philosophy of mathematics, he unwillingly finds himself trapped in a conceptual confusion when he declares his full support for Brouwer's philosophical ideas or when he labels his research as a 'Kantian research program'. In fact, as previously stated, not only Dehaene's research does not provide an empirical basis to the ideas of Kant and Brouwer but it even challenges what the two philosophers proposed.
This mistake is due to the lack of philosophical attention on the results of his experiments, which highlighted the existence of two types of mathematics, one exact and the other approximate, which in its turn shed the foundations for the study of a natural mathematics that is not based on symbols and rules. A way of doing mathematics to which Dehaene subscribes but that is completely different to the one supported by Kant and Brouwer.

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